Pricing Options Embedded in Corporate Bonds Using the Binomial Method

Qi Liu[*♣]

[♣]National University of Singapore, Singapore

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ABSTRACT: It is common for a corporate bond to include a call provision that gives the issuing company an option to call, or redeem, the bond at some prespecified set of call prices before the stated maturity date. Since the option is embedded in the bond, it is not traded publicly and thus its value is unknown to bondholders. This study is aimed to price these embedded options and their related bonds, both callable and noncallable, using the binomial method such that the method is set up to approximate the evolution of the short rate. Using reasonable values for the relevant factors and parameters, our results show that the prices of the options and the two types of bonds are noticeably affected by such factors and parameters as the maturity of the bonds, the coupon, the call price, the volatility of the short rate, and the initial short rate.

JEL classification: D92, E43, G12.

Keywords: Embedded Option, Callable Bond, Noncallable Bond, Binomial Method, Short Rate, Geometric Brownian Motion.

1 Introduction

When a company plans to raise capital to fund new investment projects or to pay off its existing loans, it usually does so by issuing debt securities that are called corporate bonds

[*]Corresponding Author. E-mail: qiliu67@u.nus.edu

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(Fabozzi, 2015; Johnson, 2010). With the help of an investment bank to underwrite and market, corporate bonds are typically issued with a face value of $1,000 and a standard coupon structure. In the United States, bondholders receive each coupon payment every six months from the issuer of a bond before its maturity, at which point they receive the face value of the bond and the last coupon payment.

Many corporate bonds contain a call provision (Brennan and Schwartz, 1977; Tuckman and Serrat, 2011) which permits the issuer to call, or redeem, part or all of the bond issue at a call price, which is generally above its face value, over a specific period before maturity. The difference between the call price and the face value is the call premium. Corporate bonds with such a feature are said to be callable. That is, a callable bond permits the issuer to pay off their debt early. Hence, the issuer may choose to redeem their bond if interest rates become lower, which allows them to re-borrow at a lower interest rate. We use the following example to illustrate a callable bond.

Suppose technology giant Google plans to borrow $10 billion and issues a 5-percent coupon bond with a maturity of 10 years and a call premium of $20. Accordingly, it pays its bondholders 5% · $10 billion or $500,000,000 in annual coupon payments. Five years after the bond issuance, interest rates fall by 200 basis points to three percent, prompting it to redeem the bonds. Under the terms of the bond contract, Google has to pay bondholders $20 premium in addition to the face value of $1,000. In total, Google pays its bondholders $10.2 billion, which is obtained through a new bond issue with a 3-percent coupon rate and a principal sum of $10.2 billion, reducing its annual coupon payments to 3% · $10.2 billion or $306,000,000.

A callable bond is composed of two parts: a standard noncallable bond and a call option embedded in the bond. That is, the holder of a callable bond enters into two transactions with the issuer. One is that she purchases a noncallable bond for which she pays a certain price. The other is that she sells a call option for which she receives the option price. Hence, the price of a callable bond is equal to the price of a comparable noncallable bond minus the price of the option (Bartter and Rendleman, 1979; Fabozzi, 2015). In other words, the difference between the prices of the noncallable bond and the callable bond is the price of the option. This option entitles the issuer to redeem the bond at certain points in time before its maturity. As the above example illustrates, when interest rates fall from five percent to three percent, it is advantageous for Google to redeem the bonds. Conversely, if interest rates remain at or over five percent, Google will do nothing because it is not to its advantage to redeem the bonds.

This research is about pricing these embedded options and their related bonds. Since the option is embedded in the bond, it is not traded publicly and thus its value is unknown to bondholders. This study is intended to price, under a variety of realistic settings, these embedded options and the two types of bonds, callable and noncallable. At this point, a proper question to ask is: What are the underlying forces that drive bond price? In their two influential papers, Black et al. (1990); Black and Karasinski (1991) point out that "the short rate is the force whose changes drive all [interest-rate-sensitive] security prices". In fact, many popular interest-rate models (Brennan and Schwartz, 1980; Cox et al., 1985; Merton, 1973; Vasicek, 1977) have indicated that the short rate is the key force that moves the prices of all interest-rate-related financial products. In this regard, we employ the following geometric Brownian motion (Black and Scholes, 1973; Hull, 2018; Rendleman and Bartter, 1980) to characterize the evolution of the short rate r:

$$dr = mrdt + \sigma rdB_t \tag{1}$$

where m - expected growth rate, σ - volatility, and $B(t)$ - standard Brownian motion.

This research contributes to the related literature in two ways. One, unlike many other studies which focus on publicly traded options, whose prices are known in real time to investors, this study prices options embedded in corporate bonds, whose prices are unknown to bondholders. Two, given that callable bonds have a call provision which allows the issuer to redeem the bonds at some discrete points in time before maturity, no exact analytic formula is available to price these options, this study proposes to employ the binomial method to approximate discretely the evolution of the short rate in Equation 1 and, accordingly, to price the options and the two types of bonds.

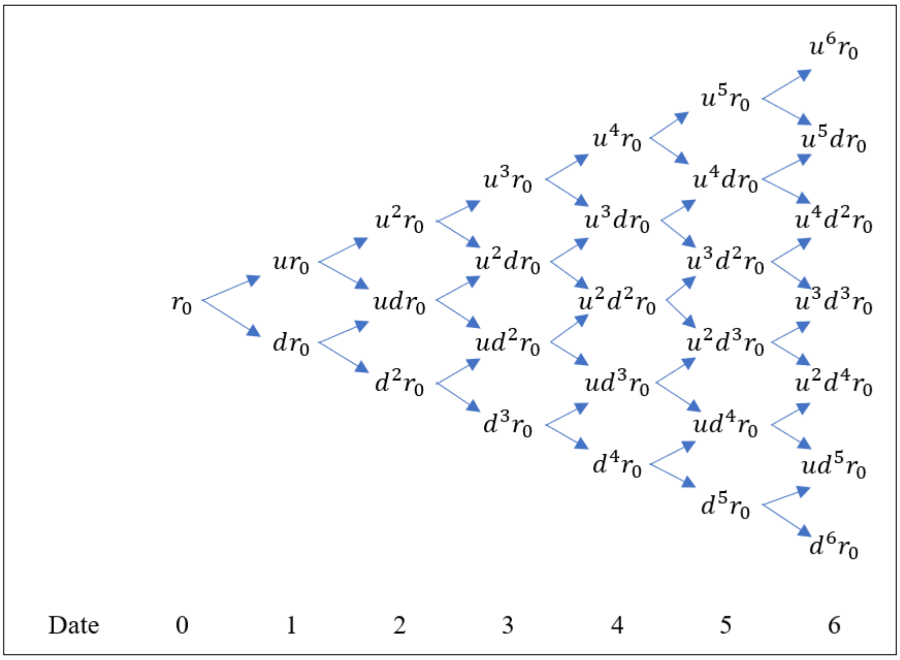
The rest of this paper will proceed as follows. In Section 2, we explain how to use the binomial method to approximate the geometric Brownian motion. In Section 3, we use a simple example to illustrate how to price an embedded option and the two types of related bonds. Section 4 specifies our research design and presents our results for the prices of the options and the two types of bonds. Section 5 concludes this study.

2 The binomial approximation to geometric Brownian motion

The crux of the binomial method is the simulation of a continuous-time model for asset price by a discrete-time model, which incorporates a risk-neutrality mechanism for making complicated computations simpler. As the time between successive time steps tends to zero, the limit of the discrete-time model will converge to the continuous-time model. In what follows, we use a binomial model as a discrete-time representation of the continuous-time geometric Brownian motion in Equation 1.

Suppose the short rate is r_0 at time 0. Under the binomial model, the short rate over a period Δt will move up to ur_0 with probability p and down to dr_0 with probability $q = 1 - p$, where u, d, and p are parameters such that $u > 1$, $d < 1$, and $0 < p < 1$. Figure 1 shows a binomial tree for the short rate in a risk-neutral world over six periods. Note that the movement of the short rate is designed in a way such that an up movement followed by a down movement leads to the same short rate as a down movement followed by an up movement. Hence, we have two rates at Date 1, three rates at Date 2, four rates at Date 3, five rates at Date 4, six rates at Date 5, and seven rates at Date 6. For ease of reference, we use state 1, state 2, and so forth to refer to each of the locations, from the bottom up, at each date. For example, in Figure 1, we have dr_0 in state 1 and ur_0 in state 2 at Date 1; d^2r_0 in state 1, udr_0 in state 2, u^2r_0 in state 3 at Date 2; and so forth.

Figure 1: Binomial representation of the movement of short rate



Using the geometric Brownian motion in Equation 1 to characterize the short rate r, we have that the expected value (Aitchison and Brown, 1957) of r after Δt is $E(r) = re^{m\Delta t}$ and the variance is $var(r) = r^2e^{2m\Delta t}\left[e^{\sigma^2\Delta t} - 1\right]$. Given the fact that e^y can be expressed as $1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \ldots$, we have $e^{2m\Delta t} = 1 + 2m\Delta t + o(\Delta t)$ and $e^{\sigma^2\Delta t} = 1 + \sigma^2\Delta t + o(\Delta t)$, where $o(\Delta t)$ is an error term which is negligible as Δt becomes small. Substituting these expressions into $var(r)$, we have $var(r) = r^2[1 + 2m\Delta t + o(\Delta t)][\sigma^2\Delta t + o(\Delta t)] = r^2\sigma^2\Delta t + o(\Delta t)$. That is, we approximate the variance of the short rate with $r^2\sigma^2\Delta t$.

We now impose three constraints such that the parameters u, d, and p for the binomial model will be chosen so that the expected value of the short rate is $re^{m\Delta t}$ and its variance is $r^2\sigma^2\Delta t$. Given that the short rate over Δt will move up to ur with probability p and down to dr with probability $1-p$, we have $E(r) = pur + (1-p)\,dr$. Hence, the first constraint for $E(r)$ is:

$$pur + (1-p)\,dr = re^{m\Delta t} \tag{2}$$

We know the variance of any random variable x is $var(x) = E[x - E(x)]^2 = E(x^2) - [E(x)]^2$. Expressing in terms of r, we have $E(r^2) = pu^2r^2 + (1-p)d^2r^2$ and $[E(r)]^2 = [pur + (1-p)\,dr]^2$. Hence, the second constraint for $var(r)$ is:

$$\left[pu^2r^2 + (1-p)d^2r^2\right] - \left[pur + (1-p)\,dr\right]^2 = r^2\sigma^2\Delta t \tag{3}$$

As mentioned above, the movement of the short rate is designed in a way such that an up movement followed by a down movement leads to the same short rate as a down movement followed by an up movement. Hence, we impose the third constraint such that the parameter d is given by $d = \frac{1}{u}$.

With $[E(r)]^2 = \left[re^{m\Delta t}\right]^2 = r^2e^{2m\Delta t}$, we can now write the three constraints for u, d, and p collectively as follows:

$$pu + (1-p)\,d = e^{m\Delta t} \tag{4}$$

$$pu^2 + (1-p)\,d^2 - e^{2m\Delta t} = \sigma^2\Delta t \tag{5}$$

$$d = \frac{1}{u} \tag{6}$$

Solving Equations 4, 5, and 6 simultaneously, we obtain:

$$u = e^{\sigma\sqrt{\Delta t}} \tag{7}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{8}$$

$$p = \frac{e^{m\Delta t} - d}{u - d} \tag{9}$$

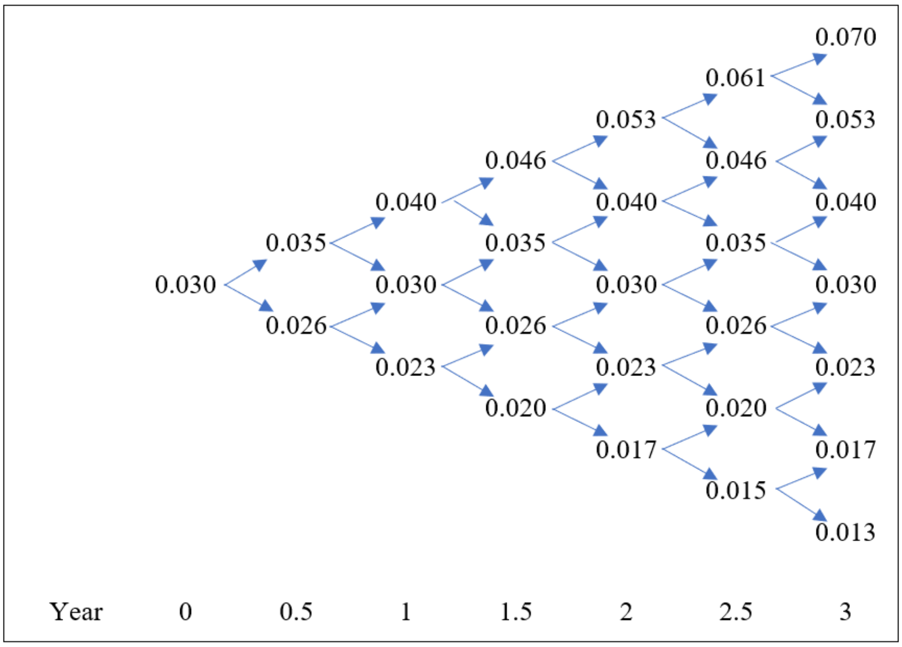
As pointed out early in this section, the binomial model contains a risk-neutrality mechanism for making complicated computations simpler. To include the mechanism into the model, we simply adjust the drift term in Equation 1 from m to (say) $\mu = m - \delta\sigma$, where δ is the market price of risk and typically assumes a negative value. Hence, we have the following new expression for p to replace Equation 9:

$$p = \frac{e^{\mu\Delta t} - d}{u - d} \tag{10}$$

3 An illustrative example

In this section, we use a simple example to illustrate how to value an embedded option and the two types of related bonds each with face value of \$100, maturity of 3 years, and coupon rate of 6 percent (which means the bonds pay out \$3 coupon every half year). In this example, the callable bond is designed such that it is callable in 1.5 years at \$101, in 2 years at \$101, in 2.5 years at \$101, and in 3 years (at maturity) at \$100. In addition, we use the following values for the relevant parameters: $\Delta t = 0.5$ year; $\mu = 0.04$; $\sigma = 0.2$; and $r_0 =$ short rate at time $0 = 0.03$. Hence, we have $u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\sqrt{0.5}} = 1.1519$, $d = e^{-\sigma\sqrt{\Delta t}} = e^{-0.2\sqrt{0.5}} = 0.8681$, $e^{\mu\Delta t} = e^{0.04\cdot 0.5} = 1.0202$, $p = \frac{e^{\mu\Delta t}-d}{u-d} = \frac{1.0202-0.8681}{1.1519-0.8681} = 0.5359$, and $q = 1 - p = 0.4641$. Figure 2 shows a binomial tree for the movement of the short rate in a risk-neutral world.

Figure 2: The binomial tree for the short rate in a risk-neutral world



Note: Each number is rounded to three decimal places for space reason.

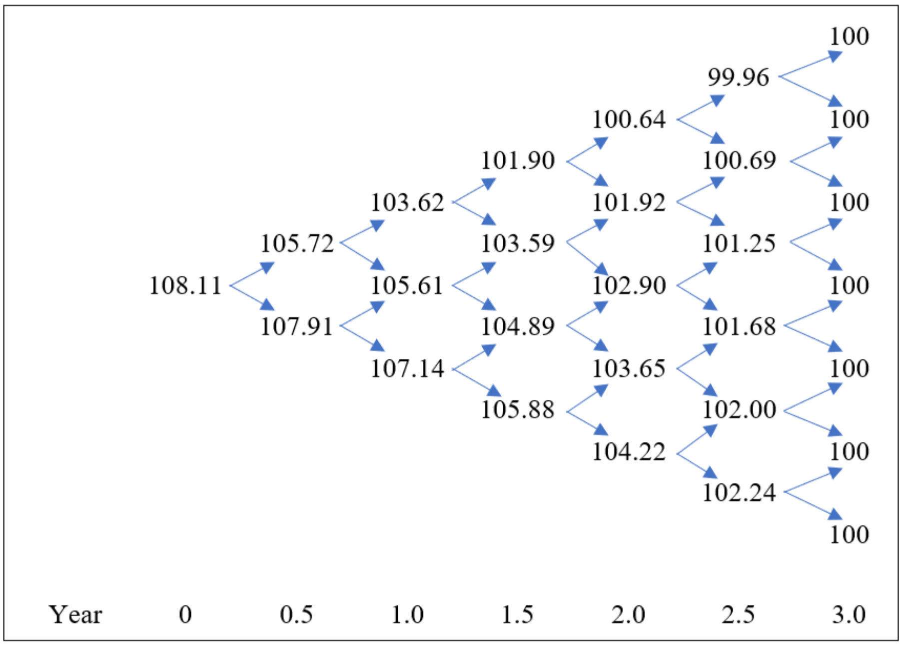
In Figure 2, the short rate begins at Year 0 with $r_0 = 0.03$. As mentioned in Section 2, the short rate is designed in a way such that an up movement followed by a down movement leads to the same short rate as a down movement followed by an up movement. For example, after one period at Year 0.5, the short rate becomes $dr_0 = 0.8681 \cdot 0.03 = 0.026$ in state 1 with a down movement and $ur_0 = 1.1519 \cdot 0.03 = 0.035$ in state 2 with an up movement; after two periods at Year 1, the short rate becomes $d^2r_0 = 0.8681^2 \cdot 0.03 = 0.023$ in state 1 with two down movements, $udr_0 = 1.1519 \cdot 0.8681 \cdot 0.03 = 0.030$ in state 2 with an up movement and a down movement, and $u^2r_0 = 1.1519^2 \cdot 0.03 = 0.04$ in state 3 with two up movements.

To value the noncallable bond, its related callable bond, and the embedded option, we use a procedure by which they are valued by starting at the end of a binomial tree and working backward. Using the short rate in Figure 2, we first construct a binomial tree for the prices of the 3-year noncallable bond. The construction begins at Year 3 where the bond value is the face value of \$100 plus the last coupon of \$3. Working backward, we calculate the bond value for each of the six states at Year 2.5. Consider state 1 at Year 2.5, which corresponds to a half-year short rate of 0.015 in Figure 2. For this state, the bond value at Year 3 is \$100 + \$3 = \$103 with an up movement or with a down movement.

Hence, the bond value in state 1 at Year 2.5 is $\frac{p\cdot\$103\ +q\cdot\$103}{1+0.5\cdot0.015}=\frac{0.5359\cdot\$103+0.4641\cdot\$103}{1+0.5\cdot0.015}=$ \$102.24. Likewise, we can calculate the bond value for all other five states at Year 2.5.

Having obtained the six bond values at Year 2.5, we move half-year backward to calculate the bond value for each of the five states at Year 2. Consider state 2 at Year 2, which corresponds to a half-year short rate of 0.023 in Figure 2. For this state at Year 2, the bond value at Year 2.5 is $\$101.68+\$3=\$104.68$ with an up movement and $\$102+\$3=\$105$ with a down movement. Therefore, the bond value in state 2 at Year 2 is $\frac{0.5359\cdot\$104.68+0.4641\cdot\$105}{1+0.5\cdot0.023}=\$103.65$. Likewise, we can calculate the bond value for each of the other four states at Year 2. Working backward through the tree in this fashion, we obtain the noncallable bond price of \$108.11 at Year 0. Figure 3 shows the binomial tree of the noncallable bond prices over the three years.

Figure 3: The 3-year tree for the prices of the noncallable bond



Note: For space reason, each number is rounded to two decimal places and the \$ sign is omitted.

With the prices of the noncallable bond in Figure 3, we now construct the 3-year tree for the prices of the call option. At Year 3 when the callable bond matures, the right to exercise the option at \$100 is not worth anything because it can be redeemed at its face value of \$100. Hence, the value of the option is zero in each of the seven states at Year 3. Before Year 3, the issuer of the callable bond has two choices whenever the option may be exercised: Exercising the option or holding the option for another half year. Let O_E be the value of exercising the option and O_H be the value of holding the option for another half year. If O_E is larger than O_H, the issuer will exercise the option and it will be worth O_E. If O_H is larger than O_E, the issuer will hold the option for another half year and it will be worth O_H. With these two choices, the issuer will choose the one that maximizes the value of the option. That is, the option is worth the maximum of O_E and

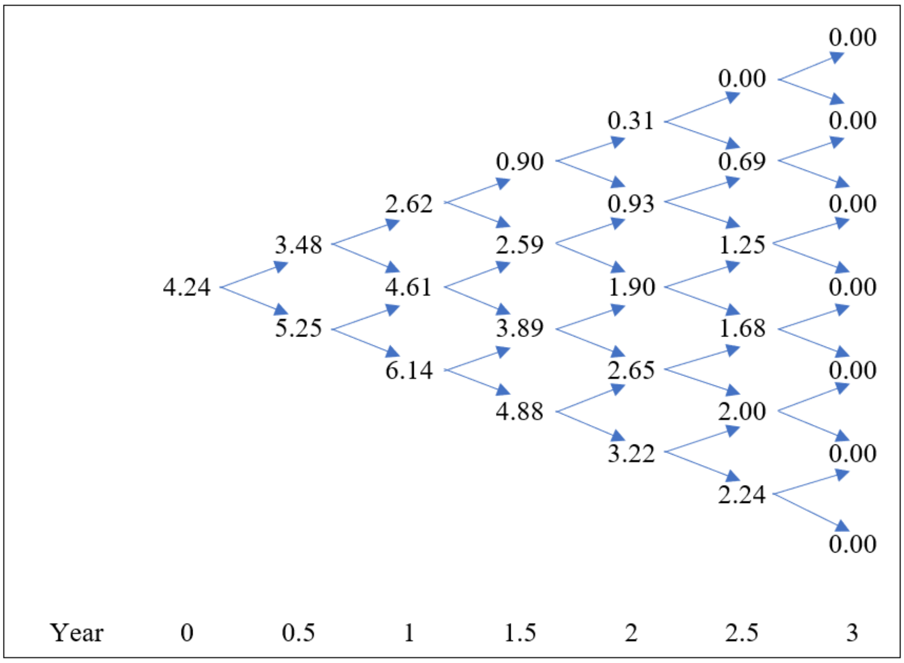
O_H, expressed simply as $\text{Max}\,(O_E,\ O_H)$.

At Year 2.5, O_H is zero because the call option will be worthless over the next half year. In states 1 through 5 at Year 2.5, the option is in-the-money and should be exercised. That is, $\text{Max}\,(O_E,\ O_H) = O_E$. But in state 6 at Year 2.5, the value of the noncallable bond is \$99.96, which is less than \$100, which means the option is out-of-the-money and should not be exercised. Hence, the option value is zero in state 6 at Year 2.5.

At Year 2, the option is in-the-money for states 1 through 3 and should be exercised. That is, $\text{Max}\,(O_E,\ O_H) = O_E$. Hence, the option value is \$3.22 in state 1, \$2.65 in state 2, and \$1.90 in state 3. However, the option is out-of-the-money and should not be exercised in states 4 and 5 at Year 2. The value of holding the option in state 4 is $O_E = \frac{p\cdot\$0.69+q\cdot\$1.25}{1+0.5\cdot0.04} = \frac{0.5359\cdot\$0.69+0.4641\cdot\$1.25}{1+0.5\cdot0.04} = \$0.93$. The value of holding the option in state 5 is $O_E = \frac{p\cdot\$0+q\cdot\$0.69}{1+0.5\cdot0.053} = \frac{0.4641\cdot\$0.69}{1+0.5\cdot0.053} = \$0.31$.

Working backward through the tree in similar fashion, we obtain the option value in each of the states at Year 1.5, Year 1, and Year 0.5. The option value at Year 0 is calculated as $\frac{p\cdot\$3.48+q\cdot\$5.25}{1+0.5\cdot0.03} = \frac{0.5359\cdot\$3.48+0.4641\cdot\$5.25}{1+0.5\cdot0.03} = \$4.24$. Figure 4 shows the 3-year tree for the prices of the embedded option.

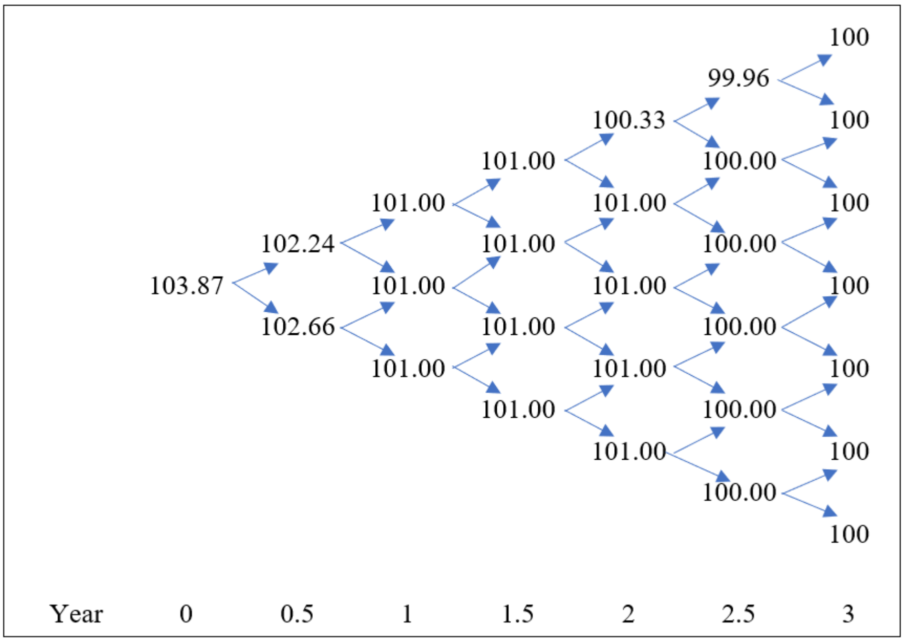
Figure 4: The 3-year tree for the prices of the embedded option



Note: For space reason, each number is rounded to two decimal places and the \$ sign is omitted.

Having obtained the tree for the prices of the option, the 3-year tree for the prices of the callable bond can simply be calculated by subtracting the option value at each node in Figure 4 from the price of the noncallable bond at the corresponding node in Figure 3. In this example, the price of the callable bond at Year 0 is \$108.11 – \$4.24 = \$103.87, where \$108.11 is the price of the noncallable bond and \$4.24 is the price of the option at Year 0. Figure 5 shows the 3-year tree for the prices of the callable bond.

Figure 5: The 3-year tree for the prices of the callable bond



Note: For space reason, each number is rounded to two decimal places and the $ sign is omitted.

4 Research design and results

4.1 Research design

To value the embedded options and the two types of related bonds each with face value of \$100, we set up our research in realistic settings. That is, we use reasonable values for the relevant factors and parameters. The factors are the time to maturity (T), the coupon payment (c) per six months, and the call price (cp). The parameters are the length of each period (Δt), the expected growth rate (μ) of the short rate, the volatility (σ) of the short rate, and the short rate (r_0) at initial time 0. In terms of factors, we consider bonds with maturity of 2, 4, 6, 8, or 10 years, coupon payment of \$3 or \$5 per six months, and call price of \$101, \$103, or \$105. In terms of parameters, we use the following values for the parameters: $\Delta t = 1$ month; $\mu = 0.04$ or 0.06; $\sigma = 0.2$ or 0.5; and $r_0 = 0.03$, 0.05, or 0.07. For example, with $\Delta t = \frac{1}{12}$ year, $\mu = 0.04$, and $\sigma = 0.2$, we have $u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\sqrt{1/12}} = 1.0594$, $d = e^{-\sigma\sqrt{\Delta t}} = e^{-0.2\cdot\sqrt{1/12}} = 0.9439$, $e^{\mu\Delta t} = e^{0.04/12} = 1.0033$, $p = \frac{e^{\mu\Delta t}-d}{u-d} = \frac{1.0033-0.9439}{1.0594-0.9439} = 0.5143$, and $q = 1 - p = 0.4857$.

One thing worth pointing out is the exact times when a callable bond is callable. In this study, we divide the time to maturity of a bond into four periods. Our design is that the bond is call-protected (which means that the bond cannot be redeemed by the issuer) for the first and fourth periods and it is callable over the second and third periods. For example, for a two-year callable bond, it is call-protected for the first and last six months; it is callable at the beginning of each month over the second and third six months.

Table 1: Prices of bonds and their embedded options for bonds with maturity of 2 years

				Noncallable Bonds			Callable Bonds			Call Options		
c	r_0	μ	σ	101	103	105	101	103	105	101	103	105
3	0.03	0.04	0.2	105.5	105.5	105.5	101.2	103.1	105.0	4.34	2.39	0.55
3	0.03	0.04	0.5	105.5	105.5	105.5	101.1	103.0	104.6	4.43	2.56	0.93
3	0.03	0.06	0.2	105.4	105.4	105.4	101.2	103.1	104.9	4.24	2.30	0.49
3	0.03	0.06	0.5	105.4	105.4	105.4	101.1	102.9	104.5	4.34	2.48	0.88
3	0.05	0.04	0.2	101.4	101.4	101.4	99.3	100.9	101.4	2.14	0.51	0.01
3	0.05	0.04	0.5	101.5	101.5	101.5	98.8	100.3	101.2	2.69	1.22	0.27
3	0.05	0.06	0.2	101.2	101.2	101.2	99.2	100.8	101.2	2.01	0.44	0.01
3	0.05	0.06	0.5	101.3	101.3	101.3	98.7	100.1	101.0	2.59	1.15	0.24
3	0.07	0.04	0.2	97.5	97.5	97.5	96.7	97.4	97.5	0.76	0.03	0.00
3	0.07	0.04	0.5	97.6	97.6	97.6	96.0	97.0	97.5	1.62	0.56	0.07
3	0.07	0.06	0.2	97.2	97.2	97.2	96.5	97.2	97.0	0.66	0.02	0.00
3	0.07	0.06	0.5	97.4	97.4	97.4	95.8	96.8	97.3	1.54	0.52	0.06
5	0.03	0.04	0.2	113.2	113.2	113.2	103.1	105.1	107.0	10.07	8.12	6.18
5	0.03	0.04	0.5	113.2	113.2	113.2	103.1	105.1	107.0	10.10	8.15	6.21
5	0.03	0.06	0.2	113.1	113.1	113.1	103.1	105.1	107.0	9.96	8.02	6.07
5	0.03	0.06	0.5	113.1	113.1	113.1	103.1	105.1	107.0	10.00	8.05	6.12
5	0.05	0.04	0.2	108.9	108.9	108.9	101.3	103.2	105.1	7.63	5.72	3.81
5	0.05	0.04	0.5	109.0	109.0	109.0	101.2	103.1	104.9	7.79	5.93	4.11
5	0.05	0.06	0.2	108.7	108.7	108.7	101.2	103.2	105.1	7.46	5.56	3.65
5	0.05	0.06	0.5	108.8	108.8	108.8	101.2	103.0	104.8	7.64	5.79	3.98
5	0.07	0.04	0.2	104.8	104.8	104.8	99.4	101.3	103.1	5.34	3.48	1.72
5	0.07	0.04	0.5	104.9	104.9	104.9	99.1	100.8	102.4	5.88	4.18	2.57
5	0.07	0.06	0.2	104.5	104.5	104.5	99.4	101.2	103.0	5.13	3.27	1.54
5	0.07	0.06	0.5	104.7	104.7	104.7	99.0	100.6	102.2	5.71	4.03	2.45

Notes: c = coupon, r_0 = short rate at time 0, μ = expected growth rate, σ = volatility. Face value of bonds is \$100 and call price is set at \$101, \$103, and \$105. The three prices for noncallable bonds are the same because the three call prices are irrelevant for these bonds.

5 Research results

Tables 1 through 5 present the prices for the embedded options and the two types of related bonds with maturities of 2, 4, 6, 8, and 10 years, respectively. We examine, one by one, how each factor or parameter affects the prices of the options and the two types of bonds. Note that when we examine a factor or parameter, it is understood that the other factors and parameters are held fixed.

In Tables 1 through 5, we observe that the longer the bond maturity (T), the larger the option price. For example, given $r_0 = 0.03$, $\mu = 0.04$, $\sigma = 0.2$, coupon = \$5, and call price = \$105, the option price is \$6.18 for $T = 2$ years, \$14.84 for $T = 4$ years, \$22.38 for $T = 6$ years, \$28.87 for $T = 8$ years, and \$34.19 for $T = 10$ years. But we do not observe this phenomenon with the noncallable or callable bonds. When the time to maturity is longer, the prices of the two types of bonds may be larger or smaller, depending on the coupon size and the short rate at time 0.

The bigger the coupon (c), the larger the option price and the prices of the two types

Table 2: Prices of bonds and their embedded options for bonds with maturity of 4 years

				Noncallable Bonds			Callable Bonds			Call Options		
c	r_0	μ	σ	101	103	105	101	103	105	101	103	105
3	0.03	0.04	0.2	110.2	110.2	110.2	102.6	104.5	106.3	7.68	5.77	3.89
3	0.03	0.04	0.5	110.5	110.5	110.5	102.0	103.8	105.5	8.47	6.71	5.02
3	0.03	0.06	0.2	109.7	109.7	109.7	102.5	104.4	106.2	7.23	5.33	3.48
3	0.03	0.06	0.5	110.0	110.0	110.0	101.9	103.6	105.3	8.14	6.40	4.75
3	0.05	0.04	0.2	102.0	102.0	102.0	99.0	100.4	101.4	2.97	1.59	0.62
3	0.05	0.04	0.5	102.7	102.7	102.7	97.4	98.9	100.1	5.26	3.81	2.57
3	0.05	0.06	0.2	101.2	101.2	101.2	98.7	100.0	100.8	2.52	1.26	0.45
3	0.05	0.06	0.5	102.0	102.0	102.0	97.0	98.4	99.6	4.93	3.54	2.36
3	0.07	0.04	0.2	94.5	94.5	94.5	93.6	94.2	94.4	0.88	0.28	0.05
3	0.07	0.04	0.5	95.6	95.6	95.6	92.2	93.3	94.2	3.38	2.28	1.40
3	0.07	0.06	0.2	93.5	93.5	93.5	92.8	93.3	93.4	0.66	0.19	0.03
3	0.07	0.06	0.5	94.7	94.7	94.7	91.6	92.6	93.5	3.12	2.08	1.25
5	0.03	0.04	0.2	125.1	125.1	125.1	106.5	108.4	110.3	18.67	16.76	14.84
5	0.03	0.04	0.5	125.4	125.4	125.4	106.4	108.3	110.2	19.05	17.16	15.28
5	0.03	0.06	0.2	124.6	124.6	124.6	106.4	108.3	110.2	18.19	16.28	14.36
5	0.03	0.06	0.5	124.9	124.9	124.9	106.3	108.2	110.1	18.62	16.74	14.86
5	0.05	0.04	0.2	116.2	116.2	116.2	103.5	105.4	107.2	12.69	10.84	8.99
5	0.05	0.04	0.5	116.9	116.9	116.9	102.9	104.6	106.3	14.07	12.33	10.64
5	0.05	0.06	0.2	115.4	115.4	115.4	103.4	105.3	107.1	11.97	10.12	8.29
5	0.05	0.06	0.5	116.2	116.2	116.2	102.7	104.4	106.0	13.53	11.82	10.15
5	0.07	0.04	0.2	108.1	108.1	108.1	100.5	102.1	103.7	7.59	5.92	4.32
5	0.07	0.04	0.5	109.3	109.3	109.3	98.8	100.3	101.8	10.46	8.93	7.47
5	0.07	0.06	0.2	107.0	107.0	107.0	100.2	101.8	103.3	6.78	5.16	3.66
5	0.07	0.06	0.5	108.3	108.3	108.3	98.4	99.9	101.3	9.92	8.43	7.01

Notes: c = coupon, r_0 = short rate at time 0, μ = expected growth rate, σ = volatility. Face value of bonds is \$100 and call price is set at \$101, \$103, and \$105. The three prices for noncallable bonds are the same because the three call prices are irrelevant for these bonds.

of bonds. For example, in Table 1 where $r_0 = 0.03$, $\mu = 0.04$, $\sigma = 0.2$, and call price = \$101, the noncallable bond price is \$105.5, the callable bond price is \$101.2, and the option price is \$4.34 when the coupon is \$3; whereas the noncallable bond price is \$113.2, the callable bond price is \$103.1, and the option price is \$10.07 when the coupon is \$5.

The bigger the call price (*cp*), the larger the callable bond price but the smaller the option price. For example, in Table 2 where $r_0 = 0.05$, $\mu = 0.06$, $\sigma = 0.5$, coupon = \$3, and the noncallable bond price[1] is unchanged at \$102.0, the callable bond price increases from \$97.0 with *cp* = 101, to \$98.4 with *cp* = 103, and to \$99.6 with *cp* = 105; whereas the option price decreases from \$4.93 with *cp* = 101, to \$3.54 with *cp* = 103, and to \$2.36 with *cp* = 105. In other words, given that the noncallable bond price is unchanged with respect to call price, there is an inverse relationship between the callable bond price and the option price because the noncallable bond price is the sum of the callable bond price and the option price.

The expected growth rate (μ) of the short rate has small detectable effect on the

[1]Note that call prices are irrelevant for noncallable bonds.

Table 3: Prices of bonds and their embedded options for bonds with maturity of 6 years

				Noncallable Bonds			Callable Bonds			Call Options		
c	r_0	μ	σ	101	103	105	101	103	105	101	103	105
3	0.03	0.04	0.2	114.2	114.2	114.2	103.8	105.7	107.5	10.38	8.54	6.72
3	0.03	0.04	0.5	115.2	115.2	115.2	102.6	104.3	106.0	12.62	10.92	9.24
3	0.03	0.06	0.2	113.0	113.0	113.0	103.7	105.5	107.2	9.36	7.55	5.80
3	0.03	0.06	0.5	114.2	114.2	114.2	102.3	103.9	105.6	11.96	10.28	8.64
3	0.05	0.04	0.2	102.0	102.0	102.0	98.2	99.5	100.4	3.76	2.52	1.55
3	0.05	0.04	0.5	104.2	104.2	104.2	96.1	97.4	98.7	8.18	6.80	5.51
3	0.05	0.06	0.2	100.3	100.3	100.3	97.4	98.4	99.2	2.94	1.88	1.08
3	0.05	0.06	0.5	102.8	102.8	102.8	95.3	96.6	97.8	7.56	6.24	5.00
3	0.07	0.04	0.2	91.3	91.3	91.3	90.1	90.7	91.0	1.16	0.61	0.28
3	0.07	0.04	0.5	94.8	94.8	94.8	89.2	90.3	91.3	5.62	4.52	3.50
3	0.07	0.06	0.2	89.2	89.2	89.2	88.4	88.8	89.1	0.80	0.39	0.16
3	0.07	0.06	0.5	93.2	93.2	93.2	88.1	89.1	90.1	5.10	4.05	3.10
5	0.03	0.04	0.2	135.9	135.9	135.9	109.7	111.6	113.5	26.15	24.26	22.38
5	0.03	0.04	0.5	136.9	136.9	136.9	109.4	111.2	113.1	27.56	25.72	23.88
5	0.03	0.06	0.2	134.6	134.6	134.6	109.6	111.5	113.4	24.99	23.11	21.22
5	0.03	0.06	0.5	135.8	135.8	135.8	109.2	111.0	112.8	26.64	24.80	22.98
5	0.05	0.04	0.2	122.2	122.2	122.2	105.7	107.4	109.2	16.57	14.79	13.03
5	0.05	0.04	0.5	124.6	124.6	124.6	104.2	105.9	107.5	20.41	18.76	17.14
5	0.05	0.06	0.2	120.4	120.4	120.4	105.4	107.2	108.9	14.97	13.22	11.49
5	0.05	0.06	0.5	123.1	123.1	123.1	103.7	105.3	106.9	19.35	17.73	16.14
5	0.07	0.04	0.2	110.3	110.3	110.3	101.0	102.5	103.9	9.26	7.76	6.36
5	0.07	0.04	0.5	114.0	114.0	114.0	98.6	100.0	101.5	15.45	14.01	12.59
5	0.07	0.06	0.2	108.0	108.0	108.0	100.3	101.7	103.0	7.72	6.35	5.07
5	0.07	0.06	0.5	112.2	112.2	112.2	97.8	99.2	100.6	14.42	13.01	11.64

Notes: c = coupon, r_0 = short rate at time 0, μ = expected growth rate, σ = volatility. Face value of bonds is \$100 and call price is set at \$101, \$103, and \$105. The three prices for noncallable bonds are the same because the three call prices are irrelevant for these bonds.

prices of the two bonds and the option. Roughly, the bigger the expected growth rate, the slightly smaller the prices of the two types of bonds and the smaller the option price. For example, in Table 3 where $r_0 = 0.07$, $\sigma = 0.2$, coupon = \$3, and call price = \$101, the noncallable bond price decreases from \$91.3 to \$89.2 and the callable bond price from \$90.1 to \$88.4 when μ increases from 0.04 to 0.06; whereas the option price decreases from \$1.16 to \$0.80 when μ increases from 0.04 to 0.06.

The larger the volatility (σ) of the short rate, the larger the noncallable bond price and the option price. For example, in Table 4 where $r_0 = 0.07$, $\mu = 0.06$, coupon = \$3, and call price = \$101, the noncallable bond price increases from \$85.0 to \$92.9 and the option price from \$0.94 to \$7.24 when σ increases from 0.2 to 0.5. Conversely, when the volatility is larger, the callable bond prices are smaller in most cases but larger in a few other cases. In Table 4, let us look at a situation where $r_0 = 0.07$, $\mu = 0.06$, and call price = \$101. When σ increases from 0.2 to 0.5, the callable bond price decreases from \$99.8 to \$97.8 with coupon = \$5 but increases from \$84.0 to \$85.7 with coupon = \$3.

The short rate (r_0) at time 0 has an obvious effect on the prices of the two bonds and

Table 4: Prices of bonds and their embedded options for bonds with maturity of 8 years

				Noncallable Bonds			Callable Bonds			Call Options		
c	r_0	μ	σ	101	103	105	101	103	105	101	103	105
3	0.03	0.04	0.2	117.6	117.6	117.6	105.0	106.8	108.5	12.60	10.82	9.09
3	0.03	0.04	0.5	120.0	120.0	120.0	103.2	104.9	106.5	16.85	15.18	13.53
3	0.03	0.06	0.2	115.5	115.5	115.5	104.6	106.3	108.0	10.84	9.14	7.50
3	0.03	0.06	0.5	118.3	118.3	118.3	102.6	104.2	105.8	15.76	14.14	12.56
3	0.05	0.04	0.2	101.6	101.6	101.6	97.1	98.2	99.2	4.46	3.37	2.42
3	0.05	0.04	0.5	106.5	106.5	106.5	95.1	96.5	97.8	11.36	10.00	8.68
3	0.05	0.06	0.2	98.7	98.7	98.7	95.5	96.4	97.1	3.26	2.36	1.61
3	0.05	0.06	0.5	104.3	104.3	104.3	93.9	95.2	96.5	10.36	9.05	7.79
3	0.07	0.04	0.2	88.3	88.3	88.3	86.8	87.3	87.7	1.48	0.96	0.59
3	0.07	0.04	0.5	95.4	95.4	95.4	87.3	88.4	89.4	8.11	6.99	5.92
3	0.07	0.06	0.2	85.0	85.0	85.0	84.0	84.4	84.6	0.94	0.58	0.33
3	0.07	0.06	0.5	92.9	92.9	92.9	85.7	86.7	87.7	7.24	6.20	5.21
5	0.03	0.04	0.2	145.4	145.4	145.4	112.9	114.7	116.6	32.58	30.72	28.87
5	0.03	0.04	0.5	148.1	148.1	148.1	112.3	114.1	115.9	35.84	34.04	32.24
5	0.03	0.06	0.2	143.1	143.1	143.1	112.7	114.5	116.4	30.44	28.59	26.74
5	0.03	0.06	0.5	146.2	146.2	146.2	111.9	113.7	115.5	34.30	32.52	30.76
5	0.05	0.04	0.2	127.1	127.1	127.1	107.5	109.2	110.9	19.58	17.88	16.20
5	0.05	0.04	0.5	132.4	132.4	132.4	105.7	107.2	108.6	26.77	25.17	23.60
5	0.05	0.06	0.2	124.0	124.0	124.0	107.1	108.7	110.3	16.91	15.25	13.66
5	0.05	0.06	0.5	130.0	130.0	130.0	104.9	106.5	108.0	25.08	23.53	21.98
5	0.07	0.04	0.2	111.7	111.7	111.7	101.2	102.5	103.8	10.59	9.26	8.00
5	0.07	0.04	0.5	119.5	119.5	119.5	98.9	100.3	101.7	20.58	19.19	17.80
5	0.07	0.06	0.2	108.1	108.1	108.1	99.8	101.0	102.1	8.30	7.10	6.00
5	0.07	0.06	0.5	116.7	116.7	116.7	97.8	99.1	100.5	18.95	17.60	16.29

Notes: c = coupon, r_0 = short rate at time 0, μ = expected growth rate, σ = volatility. Face value of bonds is \$100 and call price is set at \$101, \$103, and \$105. The three prices for noncallable bonds are the same because the three call prices are irrelevant for these bonds.

the option. In general, the larger the short rate at time 0, the smaller the prices of the two types of bonds and the option. For example, in Table 5 where, $\mu = 0.04$, $\sigma = 0.2$, coupon = \$5, and call price = \$105, the noncallable bond price is \$131.2, the callable bond price is \$112.8, and the option price is \$18.38 when r_0 is set at 0.05; whereas the noncallable bond price is \$112.9, the callable bond price is \$103.9, and the option price is \$8.96 when r_0 is set at 0.07.

6 Conclusion

When a corporation plans to raise capital to finance its investment projects, it does so by issuing corporate bonds – often with an option embedded. Unlike publicly traded options, whose prices are known in real time to investors, the prices for options embedded in corporate bonds are not known to bondholders. Hence, the usefulness of this study is that we use the binomial method to approximate discretely the evolution of the short rate in Equation 1 and, accordingly, to price the embedded options and their related bonds.

Table 5: Prices of bonds and their embedded options for bonds with maturity of 10 years

				Noncallable Bonds			Callable Bonds			Call Options		
c	r_0	μ	σ	101	103	105	101	103	105	101	103	105
3	0.03	0.04	0.2	120.4	120.4	120.4	106.2	107.9	109.5	14.20	12.52	10.86
3	0.03	0.04	0.5	125.0	125.0	125.0	104.0	105.7	107.3	20.95	19.35	17.75
3	0.03	0.06	0.2	117.2	117.2	117.2	105.5	107.1	108.6	11.65	10.06	8.52
3	0.03	0.06	0.5	122.6	122.6	122.6	103.1	104.7	106.3	19.45	17.89	16.33
3	0.05	0.04	0.2	101.0	101.0	101.0	96.1	97.1	97.9	4.91	3.93	3.04
3	0.05	0.04	0.5	109.3	109.3	109.3	94.8	96.2	97.5	14.43	13.10	11.80
3	0.05	0.06	0.2	96.8	96.8	96.8	93.5	94.3	94.9	3.33	2.57	1.91
3	0.05	0.06	0.5	106.3	106.3	106.3	93.3	94.6	95.8	13.03	11.76	10.55
3	0.07	0.04	0.2	85.6	85.6	85.6	83.9	84.3	84.7	1.67	1.21	0.84
3	0.07	0.04	0.5	96.8	96.8	96.8	86.3	87.3	88.4	10.54	9.46	8.39
3	0.07	0.06	0.2	81.1	81.1	81.1	80.1	80.4	80.6	0.98	0.68	0.44
3	0.07	0.06	0.5	93.6	93.6	93.6	84.3	85.3	86.3	9.33	8.32	7.33
5	0.03	0.04	0.2	154.1	154.1	154.1	116.2	118.0	119.9	37.84	36.01	34.19
5	0.03	0.04	0.5	159.1	159.1	159.1	115.4	117.1	118.9	43.74	41.98	40.23
5	0.03	0.06	0.2	150.4	150.4	150.4	116.0	117.8	119.6	34.46	32.64	30.83
5	0.03	0.06	0.5	156.4	156.4	156.4	114.9	116.6	118.3	41.51	39.78	38.05
5	0.05	0.04	0.2	131.2	131.2	131.2	109.7	111.3	112.8	21.55	19.95	18.38
5	0.05	0.04	0.5	140.4	140.4	140.4	107.6	109.1	110.6	32.75	31.24	29.73
5	0.05	0.06	0.2	126.5	126.5	126.5	108.8	110.3	111.8	17.75	16.25	14.76
5	0.05	0.06	0.5	137.0	137.0	137.0	106.6	108.0	109.5	30.42	28.96	27.50
5	0.07	0.04	0.2	112.9	112.9	112.9	101.6	102.8	103.9	11.27	10.09	8.96
5	0.07	0.04	0.5	125.4	125.4	125.4	99.9	101.3	102.6	25.43	24.09	22.76
5	0.07	0.06	0.2	107.7	107.7	107.7	99.4	100.4	101.4	8.32	7.31	6.35
5	0.07	0.06	0.5	121.7	121.7	121.7	98.6	99.8	101.1	23.18	21.90	20.67

Notes: c = coupon, r_0 = short rate at time 0, μ = expected growth rate, σ = volatility. Face value of bonds is \$100 and call price is set at \$101, \$103, and \$105. The three prices for noncallable bonds are the same because the three call prices are irrelevant for these bonds.

Using reasonable values for the relevant factors and parameters, our results show that the prices of the embedded options and the two types of bonds are markedly affected by such factors and parameters as the maturity of the bonds, the coupon, the call price, the volatility of the short rate, and the short rate at time 0.

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